Problem 5140. Given equilateral triangle ABC with an interior point P such that $AP = 22 + 16\sqrt{2}$, $BP = 13 + 9\sqrt{2}$, $CP = 23 + 16\sqrt{2}$. Find AB.

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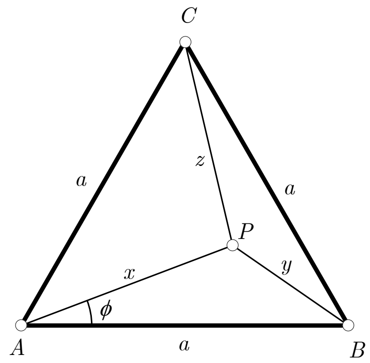

Denote $AP = x$, $BP = y$, $CP = z$, $AB = a$, $\angle PAB = \phi$. In the triangle ABP we have

$$2ax\cos\phi = x^2 + a^2 - y^2 \tag{1}$$

and

$$\Delta = [ABP] = \frac{1}{2}ax\sin\phi \tag{2}$$

The law of cosines in triangle APC yields

$$\begin{aligned} z^2 &= a^2 + x^2 - 2ax\cos(60^\circ - \phi) \\ z^2 &= a^2 + x^2 - ax(\cos\phi + \sqrt{3}\sin\phi) \\ 2z^2 &= 2a^2 + 2x^2 - 2ax\cos\phi\ - 2ax\sqrt{3}\sin\phi \end{aligned} \tag{3}$$

If we substitute (1) and (2) into (3) we get

$$\begin{aligned} 2z^2 = 2a^2 + 2x^2 - \left(x^2 + a^2 - y^2\right) - 4\sqrt{3}\Delta \qquad &\Rightarrow \\ 4\sqrt{3}\Delta = a^2 + x^2 + y^2 - 2z^2 \qquad &\Rightarrow \\ 48\Delta^2 = \left(a^2 + x^2 + y^2 - 2z^2\right)^2 & \end{aligned} \tag{4}$$

From (4), by using the Heron's formula, after some algebra, we have

$$a^4 - \left(x^2 + y^2 + z^2\right)a^2 + x^4 + y^4 + z^4 - x^2y^2 - x^2z^2 - y^2z^2 = 0$$

By putting $a^2 = t$ and bearing in mind that

$$x^2 + y^2 + z^2 = 2368 + 1674\sqrt{2}$$

$$x^4 + y^4 + z^4 - x^2y^2 - x^2z^2 - y^2z^2 = 948103 + 670410\sqrt{2}$$

we obtain the following equation:

$$t^2 - 2(1184 + 837\sqrt{2})t + 948103 + 670410\sqrt{2} = 0 \tag{5}$$

The solutions of (5) are $t = 13(17 + 12\sqrt{2})$, $t = 2147 + 1518\sqrt{2}$, from which we obtain $a = \sqrt{13}(3 + 2\sqrt{2})$ and $a = 33 + 23\sqrt{2}$. Since P is interior to the triangle ABC, it follows that $AP < a$, $BP < a$, $CP < a$. Consequently the only acceptable solution is $a = 33 + 23\sqrt{2}$. This completes the solution. $\square$